Lecture 22

Binomial Theorem and Binomial Coefficients

Binomial Theorem

Binomial coefficients

Binomial Theorem: For all non-negative integer n,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof: Can be done using induction. Skipped.

Identities on Binomial Coefficient

Theorem: For all non-negative integers n,

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \ldots + (-1)^n \binom{n}{n} = 0$$

Proof: Just put $x = -1$ and $y = 1$ in binomial theorem.

Theorem: For all non-negative integers n,

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

We will do a lot of proofs of this sort.

Proof: Just put $x = y = 1$ in binomial theorem.

Alternative Proof: Both sides are counting the number of subsets $[n]$.

Identities on Binomial Coefficient

Theorem: For all non-negative integers n and k,

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

 # $(k+1)$-element subsets of $[n+1]$ that contain $n+1$. (for $\binom{n}{k}$)

 # $(k+1)$-element subsets of $[n+1]$ that don't contain $n+1$. (for $\binom{n}{k+1}$)

 # $(k+1)$-element subsets of $[n+1]$. (for $\binom{n+1}{k+1}$)

Proof: $(k+1)$-element subsets of $[n+1]$ can be divided into two types:

- Subsets that contain $n+1$.
- Subsets that do not contain $n+1$.

Both sides of the equation are counting the same thing in different ways. ■

Pascal Triangle

nth row, starting from $n = 0$, is listing out $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 + 4 & & & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 &
\end{array}
$$

$1 + 2 + 3 + 4 = 10$

$\binom{3}{0}, \binom{3}{1}, \binom{3}{2}, \binom{3}{3}$

$$2^6 = 1 + 6 + 15 + 20 + 15 + 6 + 1$$

Identities on Binomial Coefficient

 # $(k+1)$-element subsets of $[n+1]$, where $k+2$ is the highest.

 # $(k+1)$-element subsets of $[n+1]$, where $n+1$ is the highest.

Theorem: For all non-negative integers n and k,

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \ldots + \binom{n-1}{k} + \binom{n}{k} = \binom{n+1}{k+1}$$

 # $(k+1)$-element subsets of $[n+1]$, where $k+1$ is the highest.

 # $(k+1)$-element subsets of $[n+1]$, where $k+3$ is the highest.

 # $(k+1)$-element subsets of $[n+1]$, where n is the highest.

 # $(k+1)$-element subsets of $[n+1]$.

Identities on Binomial Coefficient

Theorem: For all non-negative integers n,

$$\sum_{k=1}^{n} k\binom{n}{k} = n.2^{n-1}$$

Proof: Both sides are counting the number of ways to choose a committee among n people and a president from the committee.

On the left side, we choose k-member committees in $\binom{n}{k}$ ways, its president in k ways, and then sum all the ways.

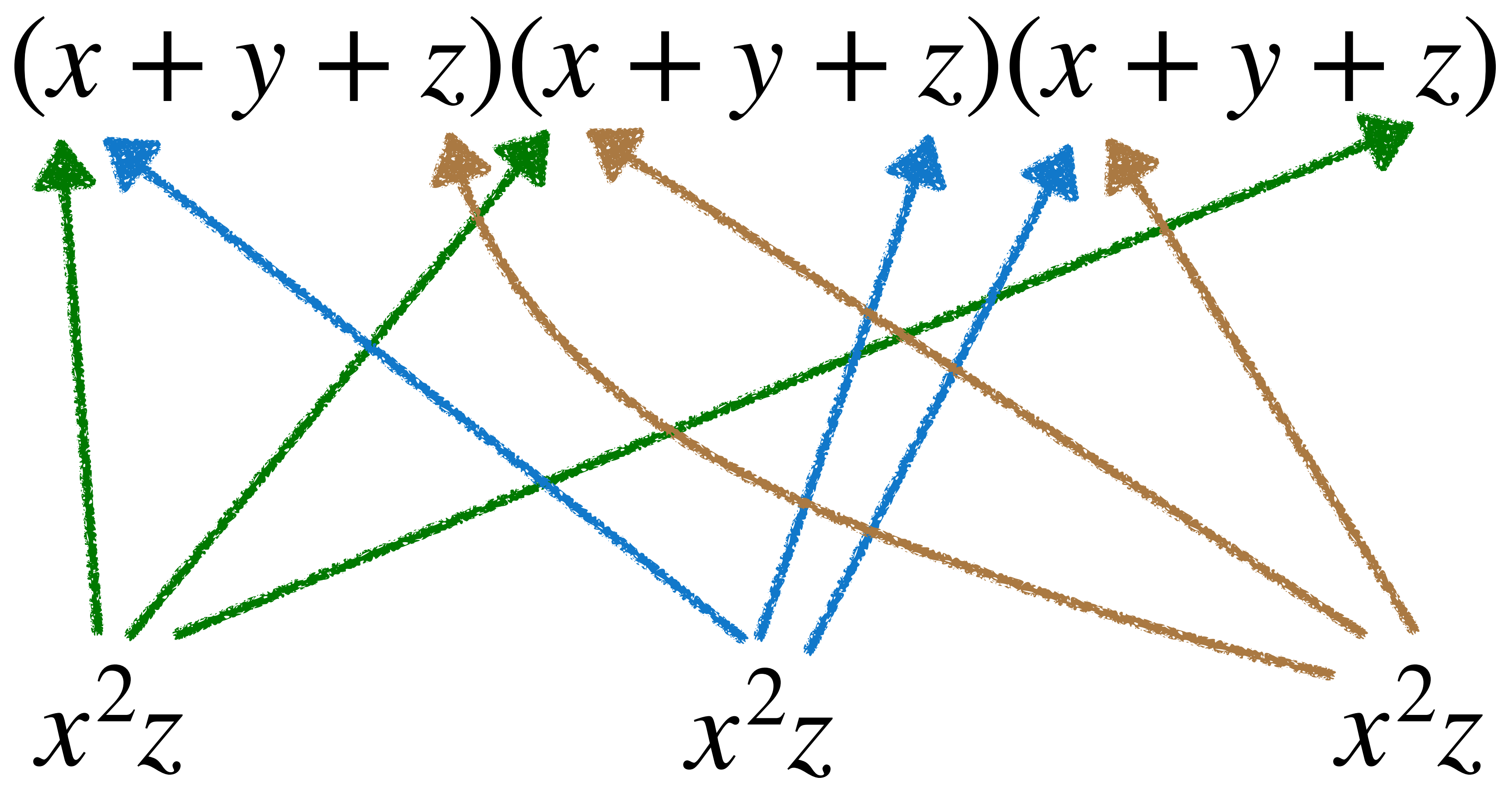
On the right side, we choose a president in n ways and then the rest of the committee in 2^{n-1}.

∎

Idea: Multinomial Theorem

$$(x+y+z)^3 = x^3 + y^3 + z^3 + 3x^2y + 3x^2z + 3y^2z + 3y^2x + 3z^2x + 3z^2y + 6xyz.$$

Why x^2z has 3 as a coefficient?



times x^2z appears in expansion $=$ # ways we can pick 2 xs and 1 z $= \dfrac{3!}{2!1!} = 3$

Multinomial Theorem

Multinomial Theorem: For all non-negative integer n and k,

$$(x_1 + x_2 + \ldots + x_k)^n = \sum_{a_1, a_2, \ldots, a_k} \binom{n}{a_1, a_2, \ldots, a_k} x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k}$$

where the sum is taken over all k-tuples of $a_1, a_2, \ldots, a_k$ such that $a_i \in \mathbb{N}$ and $n = \sum_{i=1}^{k} a_i$.